

Sum of Prime Factors

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It is a well known fact that the integers are a unique factorization domain, so that every non-zero, non-unit integer (everything but -1 , 0 , and 1) can be decomposed into a unique product of prime numbers. Of course, multiplication is not the only natural operation on the integers, we also have addition. I was thinking about prime factorizations this morning, and decided to see what would happen if I looked at the sum of each integers prime factors. For example, the number 63 has prime factorization $3 \times 3 \times 7$, so the sum of it's prime factors is $3 + 3 + 7 = 13$. Let's call the operation of adding an integers prime factors $s(z)$. I began by calculating $s(z)$ for some small values of z :

z	Factors	Sum
2	2	2
3	3	3
4	2×2	4
5	5	5
6	2×3	5
7	7	7
8	$2 \times 2 \times 2$	6
9	3×3	9
10	2×5	7
11	11	11
12	$2 \times 2 \times 3$	7
13	13	13
14	2×7	9
15	3×5	8
16	$2 \times 2 \times 2 \times 2$	8
17	17	17
18	$2 \times 3 \times 3$	8
19	19	19
20	$2 \times 2 \times 5$	9
21	3×7	10
22	2×11	13

It was pretty clear that this sequence would diverge to infinity, and I didn't

bother to rigorously prove it (it appears convergence is rather slow, consider the powers of two!). To try and cut things down, I decided to look at a similar sequence: $\frac{s(z)}{z}$. Let's add that to the table above:

z	Factors	Sum	$\frac{s(z)}{z}$
2	2	2	1
3	3	3	1
4	2×2	4	1
5	5	5	1
6	2×3	5	5/6
7	7	7	1
8	$2 \times 2 \times 2$	6	3/4
9	3×3	6	2/3
10	2×5	7	7/10
11	11	11	1
12	$2 \times 2 \times 3$	7	7/12
13	13	13	1
14	2×7	9	9/14
15	3×5	8	8/15
16	$2 \times 2 \times 2 \times 2$	8	1/2
17	17	17	1
18	$2 \times 3 \times 3$	8	4/9
19	19	19	1
20	$2 \times 2 \times 5$	9	9/20
21	3×7	10	10/21
22	2×11	13	13/22

Of course, this sequence is also doomed to diverge, owing to the primes, all of which take a value of 1. We already know that there are infinitely many primes, and also infinitely many composites (doubles of the primes, for example), which do not take a value of 1 (except for oddball 4). Thus, the sequence has no limit. The subsequences, however, are more interesting.

To fix a bit of notation; for $n \geq 2$, let $a_n = \frac{s(n)}{n}$. We will prove two theorems about subsequences of a_n :

Theorem 1: Denote the set of limits for convergent subsequences of a_n by L . $L = \{\frac{1}{m} | m \in \mathbb{Z}^+\} \cup \{0\}$.

Proof:

We'll begin by showing that $L \supset \{\frac{1}{m} | m \in \mathbb{Z}^+\} \cup \{0\}$.

First, we'll consider the case $m = 1$. This is easy, since the prime numbers all take a value equal to 1, so that a subsequence convergent to 1 is simply the prime numbers.

Now, consider $m > 1$. We need to construct a subsequence of a_n which converges to $\frac{1}{m}$.

Lemma 1: Denote the subsequence of prime numbers by p_n , where p_n is the n^{th} prime number. Let us define new sequences p_n^m by the following:

$$p_n^m = m \times p_n \quad (1)$$

Then, $a_{p_n^m}$ is a subsequence converging to $\frac{1}{m}$.

Proof:

Denote the prime factorization of m by $m_1 \times \dots \times m_k$. Then, for each n , the prime factorization of p_n^m is simply $m_1 \times \dots \times m_k \times p_n$, so that the value of $a_{p_n^m}$ is given by:

$$a_{p_n^m} = \frac{m_1 + \dots + m_k + p_n}{p_n^m} \quad (2)$$

$$= \frac{m_1 + \dots + m_k}{p_n^m} + \frac{p_n}{p_n^m} \quad (3)$$

$$= \frac{m_1 + \dots + m_k}{p_n^m} + \frac{1}{m} \quad (4)$$

Since $m_1 + \dots + m_k$ is finite (and fixed), taking the limit as n goes to infinity proves the lemma:

$$\lim_{n \rightarrow \infty} a_{p_n^m} = \frac{1}{m} \quad (5)$$

All that remains is to demonstrate a subsequence which converges to 0. Consider the subsequence given by inputting the powers of two. For the value 2^n , the sum of prime factors is simply $2n$, and the value itself is 2^n . We then compute the limit;

$$\lim_{n \rightarrow \infty} \frac{s(2^n)}{2^n} = \lim_{n \rightarrow \infty} \frac{2n}{2^n} \quad (6)$$

$$= \lim_{n \rightarrow \infty} \frac{2}{\log(2)2^n} \quad (7)$$

$$= 0 \quad (8)$$

where we applied L'Hopital's rule to complete the computation. Thus, the subsequence converges to zero, completing the proof of the first inclusion: $L \supset \{\frac{1}{m} | m \in \mathbb{Z}^+\} \cup \{0\}$.

Let us now consider the reverse inclusion; $L \subset \{\frac{1}{m} | m \in \mathbb{Z}^+\} \cup \{0\}$. We must show that if a subsequence converges, it must converge to something in the set $\{\frac{1}{m} | m \in \mathbb{Z}^+\} \cup \{0\}$. We'll proceed by contradiction. Suppose that the subsequence a_{n_k} converges to another number. Then, for large enough k , the members

of the subsequence must be distinct from the members of all of the subsequences of the form $a_{p_n^m}$, since these all converge to elements of $\{\frac{1}{m} | m \in \mathbb{Z}^+\} \cup \{0\}$. But, by prime factorization, every integer is of the form p_n^m for some n and m , so no subsequence a_{n_k} can exist, proving the second inclusion, and thus the theorem.

Theorem 2: Every subsequence a_{n_k} of a_n contains a convergent subsubsequence $a_{n_{k_l}}$.

Proof:

The proof is similar to the proof of the second inclusion of Theorem 1. Consider an arbitrary subsequence a_{n_k} . Since every integer is of the form p_n^m for some n and m , one of the following must be true:

- a_{n_k} contains an infinite number of elements from a subsequence $a_{p_n^m}$, for some m
- a_{n_k} does not contain an infinite number of elements from any such subsequence.

In the first case, simply take all such elements from the subsequence $a_{p_n^m}$, the resultant subsubsequence must also converge to $\frac{1}{m}$. In the second case, construct a subsubsequence in the following manner:

Begin with an element from some $a_{p_n^{m_1}}$. Choose the next element to be from $a_{p_n^{m_2}}$, where $m_2 > m_1$. Iterate this process. Then, the k^{th} element of the subsubsequence is bounded above by $\frac{2}{m_k}$, and clearly as $k \rightarrow \infty$, the subsubsequence must converge to 0, finishing the proof of the theorem.

Some further questions I think would be interesting to consider (I may do so in the future):

- In the original sequence $s(z)$, can we determine how many times a particular number appears? (This seems very difficult).
- Given a particular value in $k \in s(z)$, can we determine the largest value of z which produces k
- Of the subsequences of a_n which converge to 0, which ones have finite sum?