

# Reversal of Social Preference in Plurality Voting

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Suppose that we have an election, selecting one alternative from  $M$  by plurality ranking. If a candidate decides to drop out of the race, presumably all those who would have voted for them then vote for their second place choice. It turns out, because of this, that a candidate pulling out can actually reverse the results of the election! If a plurality of people say the best vegetable is Zucchini, this presumably shouldn't change if we remind them Tomato is a fruit! A simple example shows that this unusual event is indeed possible:

Consider an election with four alternatives,  $A$ ,  $B$ ,  $C$ , and  $D$ , with 9, 8, 7, and 6 first-place votes, respectively. Thus, by the plurality method, the social preference is:

$$A \succ B \succ C \succ D \tag{1}$$

Suppose we then learn the full preferences of the voters:

Number of Voters	Preferences
9	$A \succ B \succ C \succ D$
8	$B \succ C \succ D \succ A$
7	$C \succ D \succ A \succ B$
4	$D \succ C \succ A \succ B$
2	$D \succ B \succ C \succ A$

Now, if alternative  $D$  is removed from consideration, 4 voters change their first place vote from  $D$  to  $C$ , and 2 voters change it from  $D$  to  $B$ , as indicated in the updated preferences:

Number of Voters	Preferences
9	$A \succ B \succ C$
10	$B \succ C \succ A$
11	$C \succ A \succ B$

This, by the plurality system, leads to a revised social preference of:

$$C \succ B \succ A \tag{2}$$

A complete reversal! Let's do a post-mortem on what just occurred.

What caused the reversal to happen? The key is that the voters who voted  $D$  initially were forced to switch to their second choice. Because the plurality count was so close, this had a significant impact on the outcome. By judiciously arranging the second place choices of each voter, we can actually have the removal of any of the four alternatives lead to a social preference reversal! An example preference profile is given below:

Number of Voters	Preferences
6	$A \succ D \succ B \succ C$
3	$A \succ C \succ B \succ D$
5	$B \succ D \succ A \succ C$
3	$B \succ C \succ A \succ D$
5	$C \succ D \succ A \succ B$
2	$C \succ B \succ A \succ D$
4	$D \succ C \succ A \succ B$
2	$D \succ B \succ A \succ C$

It can easily be verified that the removal of any alternative then leads to the reversal of the initial social preference. We also note that the voter's preference beyond the second choice have no bearing on the initial or reversed result. We will suppress their notation for the remainder of this section.

The method above (of correctly allocating second choices), can be generalized easily to an arbitrary number of alternatives ( $M \geq 3$ ). Let us determine conditions on the first place rankings which will determine whether a social preference reversal can occur.

Suppose we have an election with  $M$  alternatives,  $\{a_1, a_2, \dots, a_M\}$ , and each alternative  $a_i$  has  $n_i$  first place (plurality votes). For convenience, we'll assume  $n_1 > n_2 > \dots > n_m$ . We wish to determine whether there exists a set of voters' preferences which causes a social preference reversal if an arbitrary alternative  $a_i$  is removed. Note that the initial social preference is given by:

$$a_1 \succ a_2 \succ \dots \succ a_M \tag{3}$$

Let's say we decide to remove alternative  $a_i$ . We need to determine if there are enough second place votes to reverse the preference order. This will give an inequality condition on  $n_i$ .

Now,  $a_1$  currently has the most first-place votes, we shouldn't allocate any second-place votes to it (it surely wouldn't help our cause). Moving on to  $a_2$ , we need to add  $(n_1 - n_2) + 1$  votes to ensure  $a_2$  has more first-place votes than  $a_1$ . We don't want to add any more than that, as we've got other alternatives to worry about ( $a_3$  needs more votes than  $a_2$ , and so on!). To keep things as

close as possible (thus conserving votes), we should add  $(n_1 - n_3) + 2$  votes to  $a_3$ . It will then have only one more vote than  $a_2$ , which has only one more vote than  $a_1$ . Letting this process run its course, we see that in order to have enough second-place votes to allocate, we require that:

$$n_i > (M - 2)n_1 - \sum_{j=2, j \neq i}^M n_j + \sum_{k=1}^{M-2} k \quad (4)$$

There is one exception, the case in which  $i = 1$ , but the modification is not difficult, we simply start by adding no votes to  $n_2$  and then following the same process as before. The resultant inequality is:

$$n_1 > (M - 2)n_2 - \sum_{j=3}^M n_j + \sum_{k=1}^{M-2} k \quad (5)$$

While this does indeed give us the means to determine whether a social preference reversal can occur, the method requires that we check  $M$  inequalities. If  $M$  is large, this is less than fun. The following lemma will make computation much faster:

**Lemma:** If  $n_i$  satisfies inequality (4) (for  $i > 2$ ), then  $n_{i-1}$  also satisfies inequality (4). If  $n_2$  satisfies inequality (4), then  $n_1$  satisfies inequality (5).

*Proof:* Suppose that  $i > 2$ , and that  $n_i$  satisfies (4). By assumption:

$$n_i > (M - 2)n_1 - \sum_{j=2, j \neq i}^M n_j + \sum_{k=1}^{M-2} k \quad (6)$$

Adding  $n_{i-1} - n_i$  to each side, we get:

$$n_{i-1} > (M - 2)n_1 - \sum_{j=2, j \neq i-1}^M n_j + \sum_{k=1}^{M-2} k \quad (7)$$

and we see that  $n_{i-1}$  satisfies (5).

Now, if  $i = 2$ , and  $n_2$  satisfies (4) we have:

$$n_2 > (M - 2)n_1 - \sum_{j=3}^M n_j + \sum_{k=1}^{M-2} k \quad (8)$$

Because  $n_1 > n_2$ , we see:

$$n_2 > (M - 2)n_2 - \sum_{j=3}^M n_j + \sum_{k=1}^{M-2} k \quad (9)$$

and thus,

$$n_1 > (M-2)n_2 - \sum_{j=3}^M n_j + \sum_{k=1}^{M-2} k \quad (10)$$

and we see that  $n_1$  satisfies (5), completing the proof of the Lemma.

**Corollary:** If  $n_M$  satisfies (4), then  $n_1$  satisfies (5) and all other  $n_i$  satisfy (4). Thus, it is possible to have a set of voters' preferences in which the removal of any alternative causes a complete social preference reversal.

This, of course, reduces our work to checking a single inequality, concluding our exploration of the problem.