When Can a Plurality Winner Be A Condorcet Loser?

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Consider an election in with N voters and M alternatives, in which the voters rank the preferences $a_1, ..., a_M$. We wish to determine for what values of N and M is it possible for a plurality winner to also be a Condorcet loser.

An instructive example is to consider $M = 3$, as in the exercise. It's clear that the optimal situation is to minimize the number of first-place votes given the plurality winner, then having all other voters rank the plurality winner last. This will minimize the number of Condorcet victories (given the requirement of a plurality of first place votes), and maximize the number of Condorcet losses, by ensuring every other vote puts the plurality winner behind every other alternative. Further, we want to divide the remaining first-place votes as equally as possible among the alternatives, to minimize the required number of first-place votes to attain plurality.

The optimal case is when the number of voters is one more than a multiple of three. This is because we can divide the first-place votes evenly, with only one extra vote going to the plurality winner. For example, if $N = 16$, and we have alternatives $\{a_1, a_2, a_3\}$, the voting profile below creates the situation we desire:

Number of Voters	Preferences
6	$a_1 \succ a_2 \succ a_3$
5	$a_2 \succ a_3 \succ a_1$
5	$a_3 \succ a_2 \succ a_1$

We see that a_1 wins the plurality 6 to 5 to 5, but loses each Condorcet pairing 10 to 6.

Let's look at neighboring values of N. For $N = 17$, we need to add one more vote. We can't add it to either $a_2 \succ a_3 \succ a_1$ or $a_3 \succ a_2 \succ a_1$, because this would prevent a_1 from being a plurality winner. This means we must adopt the voting profile:

Number of Voters	Preferences
7	$a_1 \succ a_2 \succ a_3$
5	$a_2 \succ a_3 \succ a_1$
5	$a_3 \succ a_2 \succ a_1$

This is, of course, not optimal, given that a_1 wins the plurality by more than 1. Unfortunately, by symmetry, we can't do any better in this case. The result is not quite as unbalanced if we add yet another vote, for $N = 18$. This time, we can add it to one of the other alternatives:

Number of Voters	Preferences
7	$a_1 \succ a_2 \succ a_3$
6	$a_2 \succ a_3 \succ a_1$
5	$a_3 \succ a_2 \succ a_1$

Still not completely optimal, but this time the plurality winner only beats one of the alternatives by 2 votes.

Extrapolating these simple examples, we can come up with a generalization, covering all cases with $M = 3$. This will allow us to determine for which N it is possible to have a plurality winner be a Condorcet loser. As seen above, everything depends on the value of $N \mod 3$.

If $N \mod 3 = 1$, we use the profile:

Number of Voters	Preferences
$(N-1)/3+1$	$a_1 \succ a_2 \succ a_3$
$(N-1)/3$	$a_2 \succ a_3 \succ a_1$
$(N-1)/3$	$a_3 \succ a_2 \succ a_1$

If $N \mod 3 = 2$, we use:

Number of)Voters	Preferences
$(N-2)/3+2$	$a_1 \succ a_2 \succ a_3$
$(N-2)/3$	$a_2 \succ a_3 \succ a_1$
$(N-2)/3$	$a_3 \succ a_2 \succ a_1$

And if $N \mod 3 = 0$, we require:

Number of Voters	Preferences
$N/3+1$	$a_1 \succ a_2 \succ a_3$
$N/3$	$a_2 \succ a_3 \succ a_1$
$N/3-1$	$a_3 \succ a_2 \succ a_1$

We can now calculate for which N the plurality winner is a Condorcet loser. We need the number of first-place votes for a_1 to be less than the number of last-place votes. For the $N \mod 3 = 1$ case:

$$\frac{N-1}{3} + 1 < 2\frac{N-1}{3} \tag{1}$$
$$\frac{4}{3} < \frac{N}{3} \tag{2}$$
$$4 < N \tag{3}$$

So for $N = \{7, 10, 13, ...\}$, the situation can occur.

For the $N \mod 3 = 2$ case:

$$\frac{N-2}{3} + 2 < 2\frac{N-2}{3} \tag{4}$$
$$\frac{8}{3} < \frac{N}{3} \tag{5}$$
$$8 < N \tag{6}$$

Thus, for $N = \{11, 14, 17, ...\}$, the situation can occur. Note that $N = 8$ does not permit a plurality winner which is also a Condorcet loser, even though $N = 7$ does. This explicitly shows the "disadvantage" of having $N \mod 3 = 2$.

Finally, for $N \mod 3 = 0$ we have:

$$\frac{N}{3} + 1 < \frac{N}{3} + \frac{N}{3} - 1 \tag{7}$$
$$2 < \frac{N}{3} \tag{8}$$
$$6 < N \tag{9}$$

So that for $N = \{9, 12, 15, ...\}$, the situation can also occur. Thus overall, for $N \in \{7\} \cup \{z \in Z | z \geq 9\}$, we can have plurality winner simultaneously be a Condorcet loser.

The generalization to $M \geq 3$ is not much more work. We can already see the dependance upon the value of $N \mod M$, and the patterns of preference allocation which worked for $M = 3$ generalize in a straightforward manner. Let's deal with the special cases in which $N \mod M = 0$ and $N \mod M = 1$ first.

For $N \mod M = 0$, we generalize the voting profile to:

Number of Voters	Preferences
$N/M+1$	$a_1 \succ a_2 \succ ... \succ a_M$
N/M	$a_2 \succ a_3 \succ ... \succ a_1$
...	...
$N/M-1$	$a_M \succ a_2 \succ ... \succ a_1$

Since $M-2$ alternatives receive N/M votes, and a single alternative receives one fewer, we have:

$$\frac{N}{M} + 1 < \frac{N}{M}(M-2) + \frac{N}{M} - 1 \tag{10}$$

$$2 < \frac{N}{M}(M-2) \tag{11}$$

We see that for $M = 3$, we recover the previously derived requirement of $N > 6$.

For $N \mod M = 1$, we have the voting profile:

Number of Voters	Preferences
$(N-1)/M+1$	$a_1 \succ a_2 \succ ... \succ a_M$
$(N-1)/M$	$a_2 \succ a_3 \succ ... \succ a_1$
...	...
$(N-1)/M$	$a_M \succ a_2 \succ ... \succ a_1$

To achieve a plurality winner which is a Condorcet loser, we need:

$$\frac{N-1}{M} + 1 < \frac{N-1}{M}(M-1) \tag{12}$$

$$1 < \frac{N-1}{M}(M-2) \tag{13}$$

Again, we recover the $M = 3$ result of $N > 4$.

We can then generalize when $N \mod M = C$, with $2 \geq C < M$. In this case, the plurality winner must receive $\frac{N-C}{M} + 2$ votes, and then $C-2$ alternatives must receive $\frac{N-C}{M} + 1$ votes, to take care of the remaining excess. This gives:

Number of Voters	Preferences
$(N-C)/M+2$	$a_1 \succ a_2 \succ ... \succ a_M$
$(N-C)/M+1$	$a_2 \succ a_3 \succ ... \succ a_1$
...	...
$(C+2)$ rows $\uparrow$	$(M-C+1)$ rows $\downarrow$
...	...
$(N-C)/M$	$a_M \succ a_2 \succ ... \succ a_1$

This voting profile creates the following requirement:

$$\frac{N-C}{M}+2<(\frac{N-C}{M}+1)(C-2)+(\frac{N-C}{M})(M-C+1) \tag{14}$$

$$4-C<\frac{N-C}{M}(M-2) \tag{15}$$

This gives a full generalization of which elections (characterized by number of voters N and number of alternatives M) are capable of producing a plurality winner which is also a Condorcet loser.