GRE Math Quick Review - Square Roots

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1 Introduction

In this document, we're going to go over **squares** and **square roots** of numbers. The concept of squaring a number is fairly simple:

Given a real number x, the square of x, denoted by x^2, is the number obtained by multiplying x by itself:

$$x^2 = x \times x \tag{1}$$

Some examples;

$$4^2 = 4 \times 4 = 16 \tag{2}$$

$$0^2 = 0 \times 0 = 0 \tag{3}$$

$$(-3.2)^2 = (-3.2) \times (-3.2) = 10.24 \tag{4}$$

$$(\frac{4}{11})^2 = \frac{4}{11} \times \frac{4}{11} = \frac{16}{121} \tag{5}$$

For quick reference, here are the squares of the first few integers:

x	0	1	2	3	4	5	6	7	8	9	10	11	12
x^2	0	1	4	9	16	25	36	49	64	81	100	121	144

A few things to note about squaring:

- Every real number has a square (no exceptions).
- The notation x^2 really does signify an exponential expression. Squaring is the same as using an exponent of 2.
- If $x^2 = y$, then $(-x)^2 = y$ as well.
- Except for 0, the square of any number is positive.

As you might expect, taking the square root of a number is a reverse to the process of squaring a number. Almost. Naively, we would like to define the square root of a real number x as the number which, when squared, equals the number x. Unfortunately, such a number is not unique! (In mathematics, we often say that the square root is not **well-defined**, which just means that it isn't unique). To see this, let's look at an example.

$$5^2 = 5 \times 5 = 25 \tag{6}$$

$$(-5)^2 = (-5) \times (-5) = 25 \tag{7}$$

This is no good! We have two reasonable candidates for the square root of 25. Because of this, the following notation is very important to understand:

The **principal square root** of a number x, denoted by $\sqrt{x}$, is the unique *positive* (or zero) number such that;

$$(\sqrt{x})^2 = \sqrt{x} \times \sqrt{x} = x \tag{8}$$

so that in our example above, $\sqrt{25} = 5$.

Here is a table of square roots for basic reference (it should look rather familiar to the table of squares above!);

x	0	1	4	9	16	25	36	49	64	81	100	121	144
$\sqrt{x}$	0	1	2	3	4	5	6	7	8	9	10	11	12

Finally, we note one more important fact. Since squaring a number never yields a negative result, negative numbers do not have square roots. On the GRE, if you come across the square root of a negative number, you've either made an error, or the answer choice is incorrect.

2 Essential Formulas

For the rest of the section, we'll only be dealing with principal (positive) square roots. Let's take a look at what we can do with these things.

As we saw above, the operations of squaring and square rooting are closely linked with multiplication. For this reason, we should expect nice behavior when we mix multiplication and division with square roots, but not when we mix addition or subtraction with square roots. We begin with two useful formulas:

Multiplication of Square Roots: Given two nonnegative real numbers a and b;

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b} \tag{9}$$

This is often useful when dealing with square roots that don't have nice integer values;

$$\sqrt{27} \times \sqrt{3} = \sqrt{27 \times 3} \tag{10}$$
$$= \sqrt{81} \tag{11}$$
$$= 9 \tag{12}$$

Division of Square Roots: Similarly, given two nonnegative real numbers a and b ($b \neq 0$);

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \tag{13}$$

We can utilize the rule in the same fashion;

$$\frac{\sqrt{600}}{\sqrt{24}} = \sqrt{\frac{600}{24}} \tag{14}$$
$$= \sqrt{25} \tag{15}$$
$$= 5 \tag{16}$$

Note that we get no such rules for addition of square roots. Take a look at the following counterexample;

$$\sqrt{9} + \sqrt{16} = 3 + 4 = 7 \tag{17}$$

$$\sqrt{9 + 16} = \sqrt{25} = 5 \tag{18}$$

3 Facts of Use

- The principal square root of a number can not be negative.
- Square roots of negative numbers do not exist/are undefined.
- If $a > b$, then $\sqrt{a} > \sqrt{b}$.
- The square root is also denoted by $x^{1/2}$, an exponential expression.

4 Specific Techniques

When working problems involving square roots, the following techniques are often useful:

- Estimate square roots by using common roots as "signposts." For example, if asked to use $\sqrt{26}$, note that 26 is just a little bigger than 25, and that $\sqrt{25} = 5$. This means $\sqrt{26}$ will be just a little bigger than 5.

- Simplify square roots by using the multiplication rules backwards. This is also known as **simplifying the radical** (radical is another name for the square root symbol). Take a look at the example below;

Suppose we want to simplify $\frac{\sqrt{80}}{\sqrt{45}}$. We could try applying the division rule:

$$\frac{\sqrt{80}}{\sqrt{45}} = \sqrt{\frac{80}{45}} \tag{19}$$

unfortunately this leaves us with a nasty fraction (though we could solve it from here). Instead, let's rewrite the numerator and denominator using the multiplication rule in reverse:

$$\sqrt{80} = \sqrt{16 \times 5} = \sqrt{16} \times \sqrt{5} \tag{20}$$

$$\sqrt{45} = \sqrt{9 \times 5} = \sqrt{9} \times \sqrt{5} \tag{21}$$

That sure is nice, as is allows us to simplify our original expression as follows;

$$\frac{\sqrt{80}}{\sqrt{45}} = \frac{\sqrt{16} \times \sqrt{5}}{\sqrt{9} \times \sqrt{5}} \tag{22}$$

$$= \frac{\sqrt{16}}{\sqrt{9}} \tag{23}$$

$$= \frac{4}{3} \tag{24}$$

This process is particularly important, because answers on the GRE are almost always expressed in simplest terms.

5 Exercises

1. Which of the following is closest to the value of $\sqrt{101}+\sqrt{17}$?

(A) 11 (B) 13 (C) 14 (D) 16 (E) 17

2. Column A: $\frac{\sqrt{36}}{\sqrt{64}}$ Column B: $\frac{\sqrt{49}}{\sqrt{100}}$

(A) The quantity in column A is greater. (B) The quantity in column B is greater. (C) The two quantities are equal. (D) The relationship cannot be determined from the information given.

3. Column A: $\sqrt{z^{16}}$ Column B: $(z^2)^4$

(A) The quantity in column A is greater. (B) The quantity in column B is greater. (C) The two quantities are equal. (D) The relationship cannot be determined from the information given.

4. Simplify the following expression: $\frac{\sqrt{75}}{\sqrt{50}}$.

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{\sqrt{3}}{\sqrt{2}}$ (C) $\frac{\sqrt{5}}{\sqrt{2}}$ (D) $\frac{3}{2}$ (E) $3\sqrt{2}$

5. Given that $x^2 = 121$:

Column A: $\sqrt{121}$ Column B: x

(A) The quantity in column A is greater. (B) The quantity in column B is greater. (C) The two quantities are equal. (D) The relationship cannot be determined from the information given.

6 Solutions and Explanations

1. Given that we don't have the convenience of a calculator on the GRE, it should be clear that this is an estimation question. Remember that we can not combine the square roots, since we are adding rather than multiplying. Now, 101 is just a little bit bigger than 100, and 17 is just a little bit bigger than 16, two "signpost" square roots we should know. Since $\sqrt{100} = 10$ and $\sqrt{16} = 4$, $\sqrt{101} + \sqrt{17}$ should be just a smidge larger than $10 + 4 = 14$, so the answer is **C**.

2. We could directly apply the division rule for square roots to answer this problem, but this would yield some icky fractions. Slow down for a second and notice that all the square roots in this problem have integer values (look back at the table if you don't recall these). We can simply substitute in the values to see:

$$\frac{\sqrt{36}}{\sqrt{64}} = \frac{6}{8} \tag{25}$$

$$\frac{\sqrt{49}}{\sqrt{100}} = \frac{7}{10} \tag{26}$$

This leaves us with a simple comparison of fractions. $\frac{6}{8} = .75$ and $\frac{7}{10} = .7$, so that the quantity in column A is greater, yielding answer **A**.

3. A note before solving this problem. Although no value of z is specified here, we don't have to worry about the square root being undefined. This is because z is raised to an even power, which always produces a positive (or zero) value. Now, recall from above that taking the square root of a number is the same as raising that number to the exponent of one-half. We can thus rewrite the quantity in column A as $(z^{16})^{\frac{1}{2}}$. The rest of the problem deals with exponents. We apply the rule for the exponentiation of an exponential expression:

$$(z^{16})^{\frac{1}{2}} = z^{16\times\frac{1}{2}} = z^8 \tag{27}$$

$$(z^2)^4 = z^{2\times 4} = z^8 \tag{28}$$

Thus, we see that the two quantities are equal, and the answer is **C**.

4. We wish to simplify the expression $\frac{\sqrt{75}}{\sqrt{50}}$. We'll consider two approaches. First, let's directly apply the division rule:

$$\frac{\sqrt{75}}{\sqrt{50}} = \sqrt{\frac{75}{50}} \tag{29}$$

$$= \sqrt{\frac{3}{2}} \tag{30}$$

This almost looks like answer B, and it is, in fact, the same thing. To see this, we can apply the division rule in reverse:

$$\sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}} \tag{31}$$

Another approach is to simplify the numerator and denominator separately by using the multiplication rule:

$$\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} \tag{32}$$

$$\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} \tag{33}$$

Plugging these results in, we have:

$$\frac{\sqrt{75}}{\sqrt{50}} = \frac{\sqrt{25} \times \sqrt{3}}{\sqrt{25} \times \sqrt{2}} \tag{34}$$

$$= \frac{\sqrt{3}}{\sqrt{2}} \tag{35}$$

In either case, we see that the answer is **B**.

5. This question is testing our understanding of what the principal square root is. There are two solutions to the equation $x^2 = 121$, $x = 11$ and $x = -11$. Remember that we said both of these numbers are square roots of 121. However, when we write $\sqrt{121}$, as in column A, we *always* mean the principal square root, which can not be negative. Thus, $\sqrt{121} = 11$. Now, since the quantity in column B could be either 11 or -11, we can not determine the relationship from the information given, answer **D**.