GRE Math Quick Review - Fractions

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1 Introduction

Fractions are a part of math that you're probably quite used to seeing in your daily lives. We have coupons for "half-off" at a pizza shop, we read measurements for furniture that read $84\frac{3}{8}$ inches, and we add a $\frac{1}{4}$ cup of butter to the cake batter. Now, we'll have to do some slightly heavier lifting with fractions to have success on the GRE, but it is nice to have some familiarity with the objects we're dealing with. Here, we'll cover all the required methods for manipulating fractions on the exam.

What exactly is a fraction anyway? You might be inclined to say that it represents a portion of a whole. While this is often true, it is an incomplete notion. Consider fractions like $\frac{22}{7}$, which are greater than "a whole." A better, possibly easier, interpretation is that fractions are just another way of representing division. The fraction $\frac{3}{4}$ really just means, "3 divided by 4."

Let's go over some notation. Given your everyday fraction, like:

$$\frac{124}{380} \tag{1}$$

We call that line in the middle the **fraction bar** (don't worry, you'll never be asked to identify that ever again). The number on top of the fraction bar is called the **numerator**, and the number below the fraction bar is called the **denominator**. Our interpretation of a fraction is that we divide the numerator by the denominator. Now, since this is a division problem, what can't the denominator be?

Zero. Recall that division by zero is undefined, and thus a fraction with a denominator of zero is also undefined.

Now, in the case above, the numerator is less than the denominator, which we call a **proper** fraction. A proper fraction has a value less than 1. If the numerator is greater than or equal to the denominator (and thus the value is greater than or equal to 1), we call the fraction an **improper** fraction.

Although they sure don't look like it, integers can be written as fractions as well. For example, the number 6 is the same as "6 divided by 1," which yields the fraction $\frac{6}{1}$. We can also mix integers and fractions, as in $3\frac{1}{5}$, which means $3 + \frac{1}{5}$. Not surprisingly, these numbers are called **mixed** numbers. Any mixed number can be rewritten as an improper fraction.

One last fraction style is the nasty **complex** fraction, which contains fractions in the numerator, denominator or both, such as:

$$\frac{2 + \frac{1}{4}}{\frac{7}{8}} \tag{2}$$

Fortunately, once we cover how to add, multiply, and divide fractions, we'll be able to simplify any complex fraction into a simple fraction.

2 Essential Formulas

We won't technically be using many formulas this time, as most of our work with fractions is done with straightforward, old-fashioned, hand-written calculation. We can, however, work out the basic arithmetic operations involving fractions here. Now, since fraction is simply another representation of division, we should expect the methods for multiplication and division to come out a little nicer than those for addition and subtraction, and this is indeed true.

Multiplying Fractions: Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, the product of the fractions can be obtained by simply multiplying the numerators and multiplying the denominators separately, creating a new fraction:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \tag{3}$$

This can be extended to more than two fractions. Let's look at an example:

$$\frac{2}{3} \times \frac{1}{4} \times \frac{5}{2} = \frac{2 \times 1 \times 5}{3 \times 4 \times 2} \tag{4}$$

$$= \frac{10}{24} \tag{5}$$

Since multiplying fractions is so easy, we translate division problems into multiplication ones when dealing with fractions. Think about it in a simple case; dividing a number by 2 is the same as multiplying that number by $\frac{1}{2}$ (this is the **reciprocal** of the number 2). Similarly, dividing a number by $\frac{17}{30}$ is the same as multiplying that number by $\frac{30}{17}$. We thus have the following method for division of fractions:

Dividing Fractions: Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} \tag{6}$$
$$= \frac{a \times d}{b \times c} \tag{7}$$

Let's take a concrete example:

$$\frac{3}{7} \div \frac{2}{5} = \frac{3}{7} \times \frac{5}{2} \tag{8}$$
$$= \frac{3 \times 5}{7 \times 2} \tag{9}$$
$$= \frac{15}{14} \tag{10}$$

Addition and Subtraction of Fractions: Finally, we consider the addition and subtraction of fractions. Unfortunately, it's not straightforward enough to warrant a formula, we'll have to go through a few steps. One case is rather simple; when the denominators are equal. For example:

$$\frac{1}{8} + \frac{3}{8} = \frac{4}{8} \tag{11}$$

To understand this, think of the distributive property. $\frac{1}{8}$ is the same as $1 \times \frac{1}{8}$, and $\frac{3}{8}$ is the same as $3 \times \frac{1}{8}$. Thus, the sum is $(1+3) \times \frac{1}{8}$, or $\frac{4}{8}$. It may also help to think of this visually. If we have a pizza cut into eighths, adding one eighth (one slice) to three eighths, gives four eighths.

Things are not as simple if our denominators are not equal. Consider the problem:

$$\frac{2}{11} + \frac{1}{2} = ? \tag{12}$$

If we're adding halves and elevenths, it's not clear what type of fraction the end result will be. So what is our plan of attack? Let's take advantage of what we know; how to add fractions with the same denominator. If we can rewrite the fractions we have above so that they share a **common denominator**, we'll be able to add them easily.

Recall that we can multiply the numerator and denominator of a fraction by the same number without changing its value (as this is equivalent to multiplying by 1). Let's apply this to our problem above. We multiply each fraction by the denominator of the other fraction as follows:

$$\frac{2}{11} + \frac{1}{2} = \frac{2 \times 2}{11 \times 2} + \frac{1 \times 11}{2 \times 11} \tag{13}$$

$$= \frac{4}{22} + \frac{11}{22} \tag{14}$$

$$= \frac{15}{22} \tag{15}$$

Note how the denominators of our resultant fractions became equal, so that they were easy to add together! Let's do one more example, this time with subtraction:

$$\frac{7}{10} - \frac{2}{3} = ? \tag{16}$$

Remember, first create a common denominator, then perform the subtraction:

$$\frac{7}{10} - \frac{2}{3} = \frac{7 \times 3}{10 \times 3} - \frac{2 \times 10}{3 \times 10} \tag{17}$$

$$= \frac{21}{30} - \frac{20}{30} \tag{18}$$

$$= \frac{1}{30} \tag{19}$$

Now that we've covered the basic operations, we can deal with the simplification of mixed numbers and complex fractions. It is generally a good idea to rewrite mixed numbers and complex fractions as simple fractions (though possibly improper) when working problems. The exception to this rule is when adding mixed numbers (we'll see why shortly).

Rewriting Mixed Numbers as Improper Fractions: Recall that a mixed number is really the sum of an integer and a fraction. For example:

$$4\frac{2}{5} = 4 + \frac{2}{5} \tag{20}$$

This means that rewriting a mixed number is just an addition of fractions problem (remembering that integers can be thought of as fractions with a denominator of 1):

$$4\frac{2}{5} = \frac{4}{1} + \frac{2}{5} \tag{21}$$

$$= \frac{4 \times 5}{1 \times 5} + \frac{2 \times 1}{5 \times 1} \tag{22}$$

$$= \frac{20}{5} + \frac{2}{5} \tag{23}$$

$$= \frac{22}{5} \tag{24}$$

NOTE 1: A negative mixed number, such as $-3\frac{1}{4}$ does not equal $-3 + \frac{1}{4}$! The negative sign distributes through to the fraction, so;

$$-3\frac{1}{4} = -3 - \frac{1}{4} \tag{25}$$

NOTE 2: When adding or subtracting mixed numbers, it's usually easier to leave them mixed, as the following example shows. Suppose we're trying to add:

$$12\frac{2}{3} + 9\frac{1}{5} = ? \tag{26}$$

If we were to rewrite these as improper fractions, then perform the required addition, we'd be dealing with some nasty numbers:

$$12\frac{2}{3} + 9\frac{1}{5} = \frac{38}{3} + \frac{46}{5} \tag{27}$$

$$= \frac{38 \times 5}{3 \times 5} + \frac{46 \times 3}{5 \times 3} \tag{28}$$

$$= \frac{190}{15} + \frac{138}{15} \tag{29}$$

$$= \frac{328}{15} \tag{30}$$

Yuck. That's asking for a calculation error, especially in a timed test like the GRE. Instead, let's write out the mixed numbers as sums, then add the integers and the fractions separately:

$$12\frac{2}{3} + 9\frac{1}{5} = 12 + 9 + \frac{2}{3} + \frac{1}{5} \tag{31}$$

$$= 21 + \frac{2}{3} + \frac{1}{5} \tag{32}$$

$$= 21 + \frac{2 \times 5}{3 \times 5} + \frac{1 \times 3}{5 \times 3} \tag{33}$$

$$= 21 + \frac{10}{15} + \frac{3}{15} \tag{34}$$

$$= 21 + \frac{13}{15} \tag{35}$$

Much nicer indeed. (The two answers are equal, by the way).

Simplifying Complex Fractions There are two steps to simplifying complex fractions, both of which we've already covered:

1. Rewrite the numerator and denominator as single fractions (if necessary) by addition, subtraction, multiplication, or division.
2. Perform division, dividing the numerator by the denominator as described above.

Let's try a nasty example:

$$\frac{\frac{1}{4} + \frac{3}{5}}{\frac{2}{3} - 1\frac{1}{2}} = ? \tag{36}$$

We begin by simplifying the numerator through addition:

$$\frac{1}{4}+\frac{3}{5}=\frac{1\times 5}{4\times 5}+\frac{3\times 4}{5\times 4} \tag{37}$$

$$=\frac{5}{20}+\frac{12}{20} \tag{38}$$

$$=\frac{17}{20} \tag{39}$$

Then simplify the denominator (note the mixed number in there)

$$\frac{2}{3}-1\frac{1}{2}=\frac{2}{3}-\frac{3}{2} \tag{40}$$

$$=\frac{2\times 2}{3\times 2}-\frac{3\times 3}{2\times 3} \tag{41}$$

$$=\frac{4}{6}-\frac{9}{6} \tag{42}$$

$$=\frac{-5}{6} \tag{43}$$

We then have a simple division problem:

$$\frac{17}{20}\div\frac{-5}{6}=\frac{17}{20}\times\frac{6}{-5} \tag{44}$$

$$=-\frac{17\times 6}{20\times 5} \tag{45}$$

$$=-\frac{102}{100} \tag{46}$$

So, while complex fractions can seem intimidating, simplifying them only requires the basic operations we've already learned.

3 Facts of Use

- The denominator of a fraction can never be zero (this would mean division by zero).
- Every mixed number can be written as an improper fraction.
- Every complex fraction can be simplified as a simple fraction.
- Two fractions are equal if you can multiply the numerator and denominator of one fraction by the same number to obtain the other fraction (this is equivalent to multiplying by 1).

4 Specific Techniques

When working problems containing fractions, the following techniques are often useful.

- When asked to compare fractions, write them with a common denominator (as in addition). Then the fraction with the larger numerator is the larger fraction.
- Except when adding mixed numbers, make life easier by simplifying all mixed numbers and complex fractions before doing calculations.
- Reduce all fractions to lowest terms before performing calculations. How do we reduce fractions? I'm glad you asked:

Reducing Fractions: When we were forming common denominators, we multiplied the numerator and denominator of a fraction by the same number, thus producing and equivalent fraction. In this case, the values of the numerator and denominator got larger. We can, of course, do the reverse, dividing the numerator and denominator by the same number to get an equivalent fraction with smaller values (as this is equivalent to dividing by 1. Consider the following example:

$$\frac{20000}{100000} = \frac{20000 \div 20000}{100000 \div 20000} \tag{47}$$

$$= \frac{1}{5} \tag{48}$$

Well, that's a much simpler number, yes? The only caveat is that we should divide by a number which is a **factor** of both the numerator and denominator, so as to avoid generating a complex fraction, as in the following example:

$$\frac{10}{24} = \frac{10 \div 10}{24 \div 10} \tag{49}$$

$$= \frac{1}{\frac{24}{10}} \tag{50}$$

This is less than helpful. If we instead divide by the common factor of 2, we get:

$$\frac{10}{24} = \frac{10 \div 2}{24 \div 2} \tag{51}$$

$$= \frac{5}{12} \tag{52}$$

As in the above result, if the numerator and denominator have no common factors other than 1, we call say the fraction is **in lowest terms**. This means we can reduce it no further. Being able to reduce fractions is important, not only for ease of calculation, but because the answers provided on the GRE are always given in lowest terms.

Now, reducing can always be done on lone fractions. It can not be done across the operations of addition, subtraction, or division. We can, however, reduce across multiplication, often simplifying the multiplication. It is advisable to reduce before you multiply, as if we multiply first, we'll be dealing with larger numbers, which may be more difficult to reduce. Take the following example;

$$\frac{4}{9} \times \frac{3}{8} \tag{53}$$

While neither fraction could be reduced on its own, we can reduce across the multiplication sign in two steps:

$$\begin{aligned} \frac{4}{9} \times \frac{3}{8} &= \frac{1}{9} \times \frac{3}{2} && (54) \\ &= \frac{1}{3} \times \frac{1}{2} && (55) \\ &= \frac{1}{6} && (56) \end{aligned}$$

When the multiplication finally came, we dealt with no numbers larger than 3! Let us emphasize once more that this can only done with multiplication and lone fractions, not across addition, subtraction, and division.

5 Exercises

1. Simplify the following expression:

$$\frac{1+\frac{1}{4}}{\frac{2}{3}} \tag{57}$$

(A) $\frac{8}{15}$ (B) $\frac{8}{11}$ (C) 1 (D) $\frac{11}{8}$ (E) $\frac{15}{8}$

2. What fraction of an hour is 400 seconds?

(A) $\frac{1}{18}$ (B) $\frac{1}{12}$ (C) $\frac{1}{9}$ (D) $\frac{2}{9}$ (E) $6\frac{2}{3}$

3. Column A: $\frac{3}{14}$ of 24 Column B: $\frac{6}{14}$ of 45

(A) The quantity in column A is greater. (B) The quantity in column B is greater. (C) The two quantities are equal. (D) The relationship cannot be determined from the information given.

4. Column A: $\frac{11}{18} \times \frac{6}{13}$ Column B: $\frac{6}{13} \div \frac{18}{11}$

(A) The quantity in column A is greater. (B) The quantity in column B is greater. (C) The two quantities are equal. (D) The relationship cannot be determined from the information given.

5. Simplify the following expression:

$$\frac{1}{3} \times \frac{2}{4} \times \frac{3}{5} \times \frac{4}{6} \tag{58}$$

(A) $\frac{1}{120}$ (B) $\frac{1}{30}$ (C) $\frac{1}{15}$ (D) $\frac{1}{12}$ (E) $\frac{3}{20}$

6. Which set of numbers is ordered from least to greatest?

(A) $3\frac{2}{3}, \frac{16}{5}, \frac{10}{3}$ (B) $\frac{10}{3}, \frac{16}{5}, 3\frac{2}{3}$ (C) $\frac{10}{3}, 3\frac{2}{3}, \frac{16}{5}$ (D) $\frac{16}{5}, \frac{10}{3}, 3\frac{2}{3}$ (E) $\frac{16}{5}, 3\frac{2}{3}, \frac{10}{3}$

7. One day at Johnson High School, $\frac{1}{15}$ of the students are away on a field trip. Of those not on the field trip, $\frac{1}{10}$ are absent due to illness.

Column A: The number of students away on the field trip.
Column B: The number of students absent due to illness.

(A) The quantity in column A is greater. (B) The quantity in column B is greater. (C) The two quantities are equal. (D) The relationship cannot be determined from the information given.

8. For any two positive numbers a and b, define $a \odot b = \frac{a}{a+b}$.

Column A: $(a \odot b) + (b \odot a)$ Column B: 1

(A) The quantity in column A is greater. (B) The quantity in column B is greater. (C) The two quantities are equal. (D) The relationship cannot be determined from the information given.

6 Solutions and Explanations

1. We wish to simplify the complex fraction;

$$\frac{1+\frac{1}{4}}{\frac{2}{3}} \tag{59}$$

We begin by writing the numerator as a single improper fraction:

$$\begin{aligned} 1+\frac{1}{4} &= \frac{1}{1}+\frac{1}{4} && (60)\\ &= \frac{1\times 4}{1\times 4}+\frac{1\times 1}{4\times 1} && (61)\\ &= \frac{4}{4}+\frac{1}{4} && (62)\\ &= \frac{5}{4} && (63) \end{aligned}$$

We then perform the division to write the expression as a single fraction:

$$\begin{aligned} \frac{\frac{5}{4}}{\frac{2}{3}} &= \frac{5}{4}\div\frac{2}{3} && (64)\\ &= \frac{5}{4}\times\frac{3}{2} && (65)\\ &= \frac{5\times 3}{4\times 2} && (66)\\ &= \frac{15}{8} && (67) \end{aligned}$$

Leaving us with answer **E**.

2. We wish to determine what fraction of an hour 400 seconds is. Well, an hour is sixty minutes, and a minute is sixty seconds. Thus, an hour is $60\times 60 = 3600$ seconds. All that's left is to reduce the fraction $\frac{400}{3600}$. It's clear that 100 is a factor of both the numerator and denominator, and dividing through by 100 leaves $\frac{4}{36}$. Since 36 is divisible by 4, we can simply divide the numerator and denominator by 4, leaving $\frac{1}{9}$, answer **C**.

3. We wish to compare two values: $\frac{3}{14}$ of 24 and $\frac{6}{14}$ of 45. We could perform the explicit multiplication, and compare the answers (even this wouldn't be too difficult, since the denominators are already the same), but it's easier just to notice that $\frac{3}{14} < \frac{6}{14}$, and $22 < 45$. Since the values in column A are less

than those in column B, we know that column B is greater, so the answer is **B**.

4. We wish to compare $\frac{11}{18} \times \frac{6}{13}$ and $\frac{6}{13} \div \frac{18}{11}$. Again, we could perform the required operations, but there is an easier way. Let's set up the division first:

$$\frac{6}{13} \div \frac{18}{11} = \frac{6}{13} \times \frac{11}{18} \tag{68}$$

Well, that looks awfully familiar! It's the same as column A, just in reverse order. Now since multiplication is commutative ($x \times y = y \times x$, the two columns are clearly equal, answer **C**.

5. We wish to simplify the following expression:

$$\frac{1}{3} \times \frac{2}{4} \times \frac{3}{5} \times \frac{4}{6} \tag{69}$$

Now, direct multiplication, followed by reduction will yield the correct result. It is, however, much easier to reduce first, making the numbers we have to multiply significantly smaller. Notice that the numbers 3 and 4 both appear in the numerator and denominator of some fraction. We can cancel these out, leaving:

$$\frac{1}{1} \times \frac{2}{1} \times \frac{1}{5} \times \frac{1}{6} \tag{70}$$

The multiplication is then much simpler, we get $\frac{2}{5\times 6} = \frac{2}{30}$. Dividing the numerator and denominator by 2 reduces the fraction to lowest terms, $\frac{1}{15}$, which is answer **C**.

6. We wish to order the numbers $3\frac{2}{3}$, $\frac{16}{5}$, and $\frac{10}{3}$ from least to greatest. Let's begin by rewriting $3\frac{2}{3}$ as an improper fraction:

$$3\frac{2}{3} = \frac{3}{1} + \frac{2}{3} \tag{71}$$

$$= \frac{3 \times 3}{1 \times 3} + \frac{2 \times 1}{3 \times 1} \tag{72}$$

$$= \frac{9}{3} + \frac{2}{3} \tag{73}$$

$$= \frac{11}{3} \tag{74}$$

Well, that's nice, the result has the same denominator as $\frac{10}{3}$, allowing a direct comparison. Since $11 > 10$, $3\frac{2}{3} > \frac{10}{3}$. This allows us to eliminate choices A and E. Now let's compare $\frac{10}{3}$ and $\frac{16}{5}$ by writing each with the common denominator:

$$\frac{10}{3} = \frac{10 \times 5}{3 \times 5} = \frac{50}{15} \tag{75}$$

$$\frac{16}{5} = \frac{16 \times 3}{5 \times 3} = \frac{48}{15} \tag{76}$$

Then, since $50 > 48$, we have that $\frac{10}{3} > \frac{16}{5}$. This let's us eliminate choices B and C, leaving us with only **D** as the correct answer.

7. We wish to determine which fraction of students; those on the field trip or those absent due to illness; is greater. The fraction of students on the field trip is $\frac{1}{15}$. We need to then determine the fraction of students absent due to illness. Now, the key here is that $\frac{1}{10}$ of all students *not on the field trip* are absent due to illness. What fraction of students aren't on the field trip? $1 - \frac{1}{15} = \frac{14}{15}$. Thus, the fraction of all students who go to Johnson High who are absent due to illness is $\frac{1}{10} \times \frac{14}{15} = \frac{14}{150}$. Now we can make a comparison. Instead of multiplying both fractions this time, notice that simply multiplying the fraction $\frac{1}{15}$ by ten on the numerator and denominator will yield a common denominator. This gives us $\frac{10}{150}$, and we see that the quantity in column B is greater, so that the answer is **B**.

8. We wish to compare the expression $(a \bigodot b) + (b \bigodot a)$ to 1. Well, let's get a handle on what the quantity in column A is by applying the definition given in the problem.

$$(a \bigodot b) + (b \bigodot a) = \frac{a}{a+b} + \frac{b}{b+a} \tag{77}$$

Great! The two fractions have the same denominator (as addition is also commutative), so we can add them easily to get $\frac{a+b}{a+b}$, but this is just 1! Therefore, the quantities are equal, giving us answer **C**.