GRE Math Quick Review - Exponents

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1 Introduction

In this document, we're going to look at **exponents**. Exponents signify the operation of **exponentiation**, which is a natural extension of multiplication. To understand how exponentiation works, let us look back at the relationship between addition and multiplication.

We can think of multiplication as the repeated addition of a number. For example;

$$4 + 4 + 4 + 4 + 4 = 5 \times 4 \tag{1}$$

Similarly, we can consider exponentiation as the repeated multiplication of a number. For example;

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 \tag{2}$$

In the example above, the number 3 is called the **base**, and the number 6 is called the **exponent**. The base is the number which is multiplied repeatedly, and the exponent tells us how many times to perform the multiplication. We often say that the base is raised to the exponent power (above, three is raised to the sixth power). In general, the base can be any real number, as can the exponent, however on the GRE the exponent will almost always be an integer. The only important non-integer exponent you should worry about is $\frac{1}{2}$, which is simply another notation for the (positive) square root:

$$x^{1/2} = \sqrt{x} \tag{3}$$

Now, the above interpretation only really makes sense when the exponent is a positive integer (greater than zero). It's hard to comprehend what multiplying a number by itself zero times, or negative 3 times really means. Let's deal with these cases now.

Any number *except zero* raised to the zeroth power is equal to 1. The expression 0^0 is **undefined**, which makes it unlikely to appear on a GRE question.

When we have a negative exponent, the result is equal to 1 divided by the base raised to the corresponding positive exponent. For example;

$$2^{-4} = \frac{1}{2^4} \tag{4}$$

This gives us all the interpretation/definition of exponentiation we need to determine all the important properties and formulas we might need on the GRE.

2 Essential Formulas

Because exponentiation is an operation of repeated multiplication, we should expect to find simple relations for multiplying and dividing expressions with exponents, but not when adding and subtracting expressions with exponents. This is in fact true. Fortunately, we have a number of useful formulas when multiplying or dividing such expressions. We will justify each law with an example using positive integer exponents, though the results hold true for all exponents.

Multiplication Formula 1: For any base number b, and any exponents m and n;

$$b^m \times b^n = b^{m+n} \tag{5}$$

Consider the following example, which we'll expand out using repeated multiplication:

$$3^2 \times 3^3 = (3 \times 3) \times (3 \times 3 \times 3) \tag{6}$$

$$= 3 \times 3 \times 3 \times 3 \times 3 \tag{7}$$

$$= 3^5 = 3^{2+3} \tag{8}$$

Division Formula 1: For any base number b, and any exponents m and n;

$$\frac{b^m}{b^n} = b^{m-n} \tag{9}$$

Let's see this in action with another example:

$$\frac{2^7}{2^5} = \frac{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} \tag{10}$$

$$= 2 \times 2 \tag{11}$$

$$= 2^2 = 2^{7-5} \tag{12}$$

Multiplication Formula 2: For any base numbers a and b, and any exponent m;

$$a^m \times b^m = (a \times b)^m \tag{13}$$

Again, we'll use repeated multiplication. We'll also make use of the associative property of multiplication (this allows us to rearrange things);

$$4^2 \times 9^2 = (4 \times 4) \times (9 \times 9) \tag{14}$$
$$= (4 \times 9) \times (4 \times 9) \tag{15}$$
$$= (4 \times 9)^2 \tag{16}$$

Division Formula 2: For any base numbers a and b, and any exponent m;

$$\frac{a^m}{b^m} = (\frac{a}{b})^m \tag{17}$$

This example is quite similar to the last one;

$$\frac{8^3}{5^3} = \frac{8 \times 8 \times 8}{5 \times 5 \times 5} \tag{18}$$
$$= (\frac{8}{5}) \times (\frac{8}{5}) \times (\frac{8}{5}) \tag{19}$$
$$= (\frac{8}{5})^3 \tag{20}$$

Exponentiated Exponentiation!: For any base number b, and any exponents m and n;

$$(b^m)^n = b^{m \times n} \tag{21}$$

Watch the groups of numbers carefully in this last example;

$$(4^3)^2 = (4 \times 4 \times 4)^2 \tag{22}$$
$$= (4 \times 4 \times 4) \times (4 \times 4 \times 4) \tag{23}$$
$$= 4 \times 4 \times 4 \times 4 \times 4 \times 4 \tag{24}$$
$$= 4^6 = 4^{3 \times 2} \tag{25}$$

3 Facts of Use

- A number raised to the first number is always unchanged: $x^1 = x$.
- Unless the base is zero, an exponential expression is never equal to zero.
- If the base b is positive, b^m is positive for any exponent m.
- If the base b is negative, b^m is positive for even exponents, and negative for odd exponents (positive or negative).
- If the base b is greater than 1, raising it to an exponent greater than 1 yields a larger number.
- If the base b is between 0 and 1 (a fraction), raising it to an exponent greater than 1 yields a smaller number.

4 Specific Techniques

When working problems involving exponents, the following techniques are often useful:

- Expand out exponents in terms of repeated multiplication, reducing the problem to a multiplication problem.
- Express the base as an exponential expression. For example, $27^3 = (3^3)^3 = 3^9$. This is especially useful when comparing exponential expressions with different bases.

5 Exercises

1. Simplify the following expression: $\frac{(2^4)\times(2^5)}{2^7}$.

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 3 (E) 4

2. If $4^x = 16$, what is x^3?

(A) 1 (B) 2 (C) 4 (D) 8 (E) 64

3. Column A: 8^6 Column B: 2^{16}

(A) The quantity in column A is greater. (B) The quantity in column B is greater. (C) The two quantities are equal. (D) The relationship cannot be determined from the information given.

4. Column A: 17^{23} Column B: 15^{20}

(A) The quantity in column A is greater. (B) The quantity in column B is greater. (C) The two quantities are equal. (D) The relationship cannot be determined from the information given.

5. If $x = 3$, what is the value of $((x^3)^0)^2$?

(A) 0 (B) 1 (C) 27 (D) 81 (E) 729

6. Column A: x^2 Column B: x^3

(A) The quantity in column A is greater. (B) The quantity in column B is greater. (C) The two quantities are equal. (D) The relationship cannot be determined from the information given.

7. If $\frac{2^a}{2^b} = 16$, what is $b - a$?
(A) -4 (B) -2 (C) 0 (D) 4 (E) 16

8. Column A: -1 Column B: x^6

(A) The quantity in column A is greater. (B) The quantity in column B is greater. (C) The two quantities are equal. (D) The relationship cannot be determined from the information given.

6 Solutions and Explanations

1. We wish to simplify the expression $\frac{(2^4)\times(2^5)}{2^7}$. We could go about this in two ways. Direct multiplication and division, while a bit tedious, doesn't take long, and is certainly straightforward:

$$\frac{(2^4)\times(2^5)}{2^7} = \frac{2\times2\times2\times2\times2\times2\times2\times2\times2}{2\times2\times2\times2\times2\times2\times2} \quad (26)$$
$$= 2\times2 \quad (27)$$
$$= 4 \quad (28)$$

We could also use the multiplication and division formulas, noting that the base is the same in each individual expression:

$$\frac{(2^4)\times(2^5)}{2^7} = \frac{2^9}{2^7} \quad (29)$$
$$= 2^2 \quad (30)$$
$$= 4 \quad (31)$$

In either case, we get 4, which is answer **E**.

2. Given that $4^x = 16$, we wish to determine the value of x^3. Since we need to compute the value of an expression involving x, we should first determine what x is. Well, x is a number such that $4^x = 16$. We can either recall that $4^2 = 16$, or plug in a few numbers to find that the desired value is $x = 2$. We then simply plug in this value to the expression x^3, giving $2^3 = 8$, which is answer **D**.

3. We wish to compare the quantities 8^6 and 2^{16}. Let's use the technique of writing the base 8 as an exponential expression;

$$8^6 = (2^3)^6 \tag{32}$$

$$= 2^{18} \tag{33}$$

It's then much easier to see that $2^{18} > 2^{16}$, so that the quantity in column A is greater, giving answer **A**.

4. We wish to compare the quantities 17^{23} and 15^{20}. We don't even need to perform a calculation here. Notice that the base and the exponent in column A are greater than those in column B. This means we're repeatedly multiplying a larger number, more times in column A. Thus, the quantity in column A is greater, giving answer **A**.

5. Given that $x = 3$, we wish to determine the value of $((x^3)^0)^2$. In reality, the specific value of x is not important. Recall that any nonzero number raised to the zeroth power is 1. This means that $(x^3)^0 = 1$. Plugging this in greatly simplifies things! We then have $1^2 = 1$, so the answer is **B**.

6. We wish to compare the quantities x^2 and x^3, when no specific value of x is given. There are a few ways to see that the relationship can not be determined from the information given.

First, recall that if the base is negative, then x^2 is positive (since 2 is even), but x^3 is negative (since 3 is odd). Then consider the values $x = 2$ and $x = -2$. If $x = 2$, then $2^2 = 4$ and $2^3 = 8$, so that the quantity in column B is greater. However, if $x = -2$, then $-2^2 = 4$, but $-2^3 = -8$, so that the quantity in column A is greater. This means the relationship can not be determined from the information given.

A second approach is to consider the values $x = 2$ and $x = \frac{1}{2}$. As we just saw, for $x = 2$, the quantity in column B is greater. For $x = \frac{1}{2}$, $(\frac{1}{2})^2 = \frac{1}{4}$, and $(\frac{1}{2})^3 = \frac{1}{8}$, so that the quantity in column A is greater. This is because when the base number is a fraction, increasing the value of the exponent decreases the result. Thus, the relationship can not be determined from the information given.

A third approach is to consider the value $x = 2$ and $x = 1$. Again, for $x = 2$, the quantity in column B is greater. But for $x = 1$, $1^2 = 1$ and $1^3 = 1$, so the two quantities are equal. For a final time, we see that the relationship can not be determined from the information given, so the answer is **D**.

7. Given that $\frac{2^a}{2^b} = 16$, we wish to determine $b - a$. First, we need to note that 16 is equal to 2^4. Then, we can apply one of our division rules to the left hand side:

$$\frac{2^a}{2^b} = 2^{a-b} \tag{34}$$

This means that $a - b = 4$, so that $b - a = -4$, yielding answer **A**.

8. We wish to compare the quantities -1 and x^6, again, no specific value of x is given. We need to recognize that since the exponent is an even number, the quantity x^6 will be positive if x is positive *or* negative. Further, if $x = 0$, the quantity is zero. Thus, the quantity in column B is always greater than or equal to 0, to this quantity is greater than -1. The answer is **B**.