Calculus GHL 0.1 - Functions

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Suppose I knew nothing about calculus, other than that it was a type of mathematics. What would be a natural first question, a fundamental question?

- What is calculus used for?
- What kind of problems can calculus solve?
- What techniques are used?

While each of these questions is certainly pertinent, they are irrelevant without first understanding what mathematical objects we use in calculus. What objects can we apply calculus techniques to, and what objects will those techniques produce?

In general, we only work with two types of objects in standard, introductory calculus; **numbers** and **functions**. Hopefully, you are by now familiar with numbers, the "answers" we so often seek in math problems. We'll spend the rest of this GHL learning about the second object, the function.

Our first concept will be that of a **set**, which we can think of simply as a collection of objects. We'll restrict our objects to be real numbers, though in the abstract, anything can be an element of a set (complex numbers, rocks, this text, last night's pizza). There isn't much more to say formally, but since sets are so important, we'll give them a definition anyway, as well as introduce a little notation:

Definition: A **set** is a collection of real numbers. An **element** of a set is some number in the collection. Typically, we denote sets by capital letters and elements by lower case letters. If a number x is an element of a set S, we write $x \in S$ (read: x is in S). If not, we write $x \notin S$ (read: x is not in S).

Example: The following are examples of sets:

- $1, 2, 3$
- 498375
- All real numbers between 4 and 10

Let us denote that last set (all real numbers between 4 and 10) by S. Then the following are true statements:

- $5 \in S$
- $12 \notin S$
- $8 \in S$

But what about $4 \in S$ or $10 \notin S$? Our statement for S certainly is ambiguous! Do we mean to include 4 in our set? 10? These types of sets, called **intervals**, occur often in calculus, and we have a special notation to deal with them:

Intervals have two **endpoints**, and there are three types of endpoints; open, closed, or unbounded. An **open** endpoint is not included in the interval, and is denoted by a parenthesis. A **closed** endpoint is included in the interval, and is denoted by a bracket. An **unbounded** endpoint indicates an "endpoint" of ∞ or $-\infty$. Let us consider some examples:

Example:

- The interval $(0, 1)$ consists of all numbers between 0 and 1, but does not include either 0 or 1.
- The interval $[-2, 2]$ consists of all numbers between -2 and 2, including both -2 and 2.
- The interval $[3, 4)$ consists of all numbers between 3 and 4, including 3, but not 4.
- The interval $(-\infty, 0]$ consists of all numbers less than or equal to 0.
- The interval $(0, \infty)$ consists of all numbers greater than (not including) 0.

Your turn!

Write the following sets in interval notation:

1. All real numbers between 2 and 5, not including either 2 or 5.
2. All real numbers between -1 and 3, including 3 but not -1.

3. All real numbers greater than or equal to 4.
4. All real numbers.

Determine whether the following statements are true:

1. $4 \in [4, 5]$
2. $7 \in (-\infty, 0)$
3. $3 \in (1, 3)$
4. $2.2 \in [1, 4)$

Alrighty, that should have us covered in terms of sets for now. Let's use these objects to help define the objects we're looking for; functions. Perhaps you've heard an intuitive description of a function as a machine: a function "machine" receives numbers (x) as inputs, applies some rule, say $y = 3x + 4$, to produce output numbers y. This interpretation is useful to us, and it is close to a more formal definition of a function.

Definition: Consider two sets X and Y. A **function** f from X to Y (written $f : X \to Y$), is a rule which assigns to each $x \in X$ *exactly one* number $y \in Y$, denoted by $f(x)$.

Note the important statement that was not apparent in our machine "definition," that each input corresponds to exactly one output. As expected, we have a few more pieces of notation and nomenclature to deal with concerning functions:

Definition: The **domain** D of a function $f : X \to Y$ is the set of all real numbers $x \in X$ for which $f(x)$ is defined (that is, $f(x)$ is a real number). We call the number $f(x) \in Y$ (read: f of x), the **value** of the function f at x. The set of all $y \in Y$ such that $y = f(x)$ for some $x \in D$ is called the **range** of f.

This is all well and good, but what do functions actually *look* like? Often (especially in calculus), we can write a function as a formula:

$$f(x) = 3x + 4 \tag{1}$$

We often write such a function as:

$$y = 3x + 4 \tag{2}$$

and for our purposes, the two mean exactly the same thing. In these cases, we call x the **independent variable** and y the **dependent variable**, because the result for y depends on which value of x we choose. Let's now perform a full analysis on this function.

Well, first we should check that this thing is actually a function! Recall the criterion from above, that each element of the domain must be assigned exactly one element of the range. This is fairly clear here, as if we plug in a number, say 5, we get exactly one number out, in this case 19. What exactly are the domain and range? Remember that the domain is the set of all real numbers for which the function is defined, or produces another real number. No matter what number we pick, if we multiply by three and then add four, we're getting another number back, so here the domain is all real numbers, often denoted R. Similarly, any number can be written as three times another number plus four, so that the range is also all real numbers. Cool. Let's try another function, and you fill in the blanks:

$$g(t) = t^2 \tag{3}$$

An important point here. Functions do not have to be denoted by the letter f. Here we have the function g, with (gasp!) independent variable t. The letters like x and t are just variables, and we can represent them with any symbol we like. I once did an assignment using smiley-face as my variable, so don't worry if the standard f, x, and y are MIA.

Again (and this will be the last time in this lesson), let's check that g is actually a function. What is the criterion to check again?

Why is it true here?

Now let's investigate the domain and range. Clearly, it makes sense to square any real number, so that the domain of g is all real numbers (again, denoted by R). The range, however, is not all real numbers. Determine the range and write it interval notation:

So far, all of our functions have had a domain of all real numbers. What would prevent a number from being in the domain? Let's consider what "isn't allowed" in mathematics. Two things come to mind (and they take care of most domain problems we'll run into); dividing by zero and taking the square root of a negative number. We've learned in basic algebra that dividing by zero simply isn't allowed, and that taking the square root of a negative number gives an imaginary (or complex) number, rather than a real number. This means we'll have to watch out for functions trying to perform one of these "illegal operations".

Example: Find the domain of the function $f(x) = \frac{3}{\sqrt{x-5}}$. First we'll check when the denominator is equal to zero:

$$\sqrt{x-5} = 0 \tag{4}$$
$$x - 5 = 0 \tag{5}$$
$$x = 5 \tag{6}$$

Now, we have a square root, so we can only apply the function to values of x which make the argument of the square root nonnegative. The argument here is $x - 5$, which is nonnegative as long as $x \geq 5$. But recall that we can't use 5 either, since it would cause a division by zero. Thus, the domain D of the function f is all real numbers greater than 5. Write the domain in interval notation:

You try a function now: $h(x) = \frac{1}{x(x-2)}$. Determine the domain of this function. Note that there are no square roots here, so you'll only have to determine where the function tries to divide by zero.

To write this domain in interval notation, you'll need to use the **union** symbol: $\cup$. If we have two sets (or intervals), we can create a new set, called the union, which consists of all elements in either set. For example, the union $(-\infty, -2) \cup (2, \infty)$ consists of all numbers less than -2 and all numbers greater than 2. Are 2 and -2 included?

We should now take a moment to think about why functions are useful. Calculus is a science which deals with relations between quantities, rates of change, and amounts. Functions provide a concise way to describe the relations between two things. Consider a store which sells carpet at 12 dollars per yard. Denoting the number of dollars in cost by d, and the number yards purchased by y, we have a cost function: $d = 12y$. Both simple and complicated relations can be expressed in function notation, so that we can apply techniques of calculus to learn detailed information about those relations.

Additional Exercises: Compute $f(2)$, $f(c)$, $f(x+h)$ for the following functions:

1. $f(x) = x^2$
2. $f(x) = \frac{2}{x}$

Find a value of x such that $g(x) = 2$ for the following functions:

1. $g(x) = 5x - 4$
2. $g(x) = x^4 - 2$

Find the domain and range for each of the following functions, and write in interval notation:

1. $f(x) = x$
2. $g(x) = x^4$
3. $h(x) = \sqrt{x}$
4. $d(x) = \frac{\sqrt{x-1}}{x-5}$
5. $c(x) = \sqrt{x^2}$

Write down three examples of functions relating quantities in your everyday lives.

Can a function have the same value at two different points? If so, give an example, if not, explain why

Can a function have multiple values at a point x? If so, give an example, if not, explain why.